OPTIMUM MANAGEMENT OF THE NETWORK OF CITY BUS ROUTES BASED ON A STOCHASTIC DYNAMIC MODEL

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(Communicated by Kok Lay Teo)

Abstract. In this paper, we develop a stochastic dynamic model for the network of city bus routes subject to resource and other practical constraints. We define an objective function on the basis of four terms: fuel cost, operating cost, customers waiting time, and revenue of the bus company. Hereafter, an optimization problem is formulated and solved by use of nonlinear integer programming. If the technique presented here is implemented, it is expected to boost the bus company's revenue, reduce waiting time and therefore promote customer satisfaction. A series of numerical experiments is carried out and the corresponding optimization problems are addressed giving the optimal number of buses allocated to each of the bus routes in the network. Since the dynamic model proposed here can be applied to any network of bus routes, it is believed that the procedure developed in this paper is of great potential for both the city bus company and the customers.

1. **Introduction.** Public transport is regarded as a shared service that is available for the general public to use. It involves different modes, such as buses, light rail and rapid transit, etc [10]. They play an important role in regional patterns of development, economic viability, environmental impacts, and in maintaining socially acceptable levels of quality of life [7]. However, there still exist some problems with conventional public transportation, especially city buses. For instance, there are many bus routes built periodically but do not fit the current needs of the general public [6]. This results in unnecessarily massive exhaust emission causing urban pollution detrimental to public health. In order to alleviate this situation, it is necessary to cut down the number of busses in the network and rearrange the transportation system so as to maximize the movement of the public, minimize waiting time while maximize company's revenue.

Currently, the dominant mode of public transport is buses in most of the cities. Therefore, there is great potential to reduce public transfer times, traffic congestion, and exhaust emission by improving the network of city bus routes, such as rerouting the public vehicles and coordinating buses required for different routes in the network.

Based on our previous work [9] that presented the dynamic model for a single bus route and solved the optimization problem using calculus of variations, in this paper we develop the dynamic model for a network of bus routes each with multiple stations and formulate the corresponding optimization problem from system's point of view [1], [2].

To begin with, for any city we consider any one typical bus route and determine the frequency of a bus passing by each station along this specific route. It is clear that the frequency is determined by the number of buses deployed for this route. Then we introduce the mathematical expressions for the fuel cost, operating cost, customers waiting time, and the company's revenue. Following this, an appropriate objective functional is proposed in terms of all the bus routes in the network and an optimization problem is formulated with the objective of balancing the customers waiting time and the bus company's revenue. The optimization problem is solved by use of nonlinear integer programming giving the optimal strategy of allocating an appropriate number of buses for each route served by the company.

The rest of the paper is organized as follows: The statement of the problem under consideration is presented in section 2 along with the dynamic model of the network of bus routes introduced in section 3. Then the corresponding optimization problem is formulated in section 4. Section 5 demonstrates a series of numerical experiments and results with appropriate interpretation. Finally, conclusions are made in section 6.

2. **Problem statement.** It is a common knowledge that in some cities passengers (riders) have to wait for a long time in order to get into the bus they want, which usually reduces the customer satisfaction and generates negative feedback. This leads to the reduction of demand as customers may choose alternate services (such as a competing bus company). In [3], it was reported that workers in Beijing (capital of China) had average daily commute time of 52 minutes each way in 2015. This large amount of commute time can be significantly reduced if the network of bus routes is better (optimally) managed. More specifically, the city bus company can relatively increase the frequency of buses passing through some busy lines or consider rerouting some busses accordingly. More importantly, the company can put more focus on the coordination of all the different city bus routes within the network.

In some cities, however, buses may frequently pass by a number of stations without many customers to serve. This results in the waste of energy and resources for the bus company and increase of greenhouse gasses. As a result, the bus company may cancel many trips without notifying any customers who are still following the regular bus schedules. In [8], it was shown by Metro analysis that there were 4530 cancellations from OC Transpo in the city of Ottawa (capital of Canada) in 79-day period (between Feb. 2 and April 22, 2017) which included some of the worst winter weather of this past year.

It is obvious that the bus company cannot make much profit if an unnecessarily redundant number of buses is allocated to any bus route. On the contrary, if there are very few buses allocated for the route, riders may have to wait for a longer time causing customer dissatisfaction. Therefore, there exists a tradeoff between the company's revenue and customers waiting time. In order to balance these two key factors, a dynamic model of the network of city bus routes is developed and the corresponding optimization problem is formulated and solved in the following sections.

3. **Dynamic model of the network of city bus routes.** It is clear that there is a large number of bus routes in any city and that each one of them should be managed optimally at the same time. In this section, we focus on the city network including multiple bus routes and develop the appropriate dynamic model. Note that this dynamic model can be generally applied to any network of bus routes. Let us denote the total number of bus routes in any given city by n, the total number of stations along the i-th bus route by s_i, the total number of buses available to the company by M, and the number of buses allocated for the i-th route by x_i. The i-th city bus route in the network is shown in Figure 1 for demonstration.
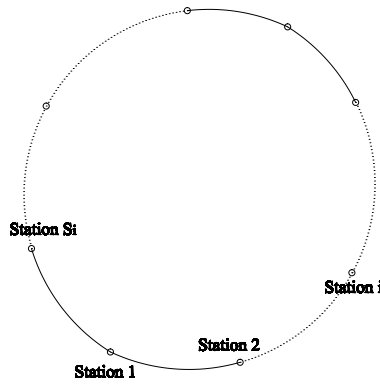

FIGURE 1. The i-th city bus route.

3.1. **Frequency of the city bus.** We start with the i-th bus route and then extend it to the whole network. It is assumed that city buses run for $24h$ for the i-th bus route. Here we do not take into account the transition time during which any bus waits in an intersection or stops at a station for customers to get on or off the bus. It is reasonable to assume that every bus passes by each station along the route at the same frequency. Thus the time interval between two consecutive passes along the i-th bus route is given by,

$$\tau_i = \frac{L_i}{x_i \cdot v_i} \tag{1}$$

where L_i is the total length of the i-th bus route, x_i is the number of buses allocated to this specific route and v_i is the average speed of the city buses running along this route. It is assumed that city buses run at an average speed v_i along the i-th route following the speed limit in the city, for example 30km/h. It is clear that the time period τ_i decreases with the increase of number of buses x_i, which implies that city buses will pass by the route more frequently thereby reducing customers waiting time.

3.2. **Fuel cost.** The fuel cost for each city bus very much depends on the speed of the bus itself whenever it is running. However, we omit the idling cost when the bus is stopping at a station or in an intersection since this only happens for a relatively short time as mentioned before. Here the acceleration doesn't necessarily need to be taken into consideration because it is assumed that every city bus runs at a relatively constant speed v_i (average speed for the i-th route) most of the time.

Therefore, the total fuel cost for all the buses serving the i-th route over the whole day is given by,

$$\hat{F}(x_i) = \sum_{k=1}^{x_i} \int_{t_0}^{t_N} q_{i,k} v_i^2 dt \tag{2}$$

where $q_{i,k}$ is the coefficient of fuel cost which is dependent on the mass and physical shape (aerodynamic) of the bus. The time period $[t_0, t_N]$ covers the whole day from midnight to midnight. Hence, the total fuel cost for all the bus routes in the network is given by the following expression,

$$F(x) = \sum_{i=1}^{n} \hat{F}(x_i) = \sum_{i=1}^{n} \sum_{k=1}^{x_i} \int_{t_0}^{t_N} q_{i,k} v_i^2 dt \tag{3}$$

where $x = (x_1, x_2, x_3, \cdots, x_n)$ denotes the vector of the number of buses allocated to each route in the network and n is the total number of routes within the network as mentioned above.

3.3. **Operating cost.** As long as a bus is running, there will be an operating cost (wage for the corresponding bus operator). Clearly, the unit payment for bus drivers varies with time. More specifically, it is higher during the night and relatively lower during the day time. Let $d_{i,k}$ denote the operating cost per unit time (wage for the k-th bus operator) for the i-th bus route. Thus, a reasonable expression for the total operating cost for the i-th route during the whole day is given by,

$$\hat{P}(x_i) = \sum_{k=1}^{x_i} \int_{t_0}^{t_N} d_{i,k}(t) dt \tag{4}$$

So the total daily operating cost for the whole network is as follows,

$$P(x) = \sum_{i=1}^{n} \hat{P}(x_i) = \sum_{i=1}^{n} \sum_{k=1}^{x_i} \int_{t_0}^{t_N} d_{i,k}(t) dt, \tag{5}$$

where $d_{i,k}(t)$ can be chosen as,

$$d_{i,k}(t) = \begin{cases} \alpha_{i,k} & \text{time between 6 AM and 8 PM,} \\ \beta_{i,k} & \text{otherwise.} \end{cases}$$

The unit payment for bus operators, $d_{i,k}(t)$, depends on time of the day or night and it is also dependent on the experience of the driver. However, for simplicity we do not take into account the experience of different bus operators. For numerical experiments, it is set lower ($\alpha_{i,k} = 50$) during the daytime: from 6:00 AM to 8:00 PM, while relatively higher ($\beta_{i,k} = 80$) during the night: from 0:00 AM to 6:00 AM and from 8:00 PM to 12:00 PM. Note that the distribution of wages over the whole day can be chosen as necessary (according to different cities).

3.4. **Customers waiting time.** At each bus station, the arrival of customers follows the nonhomogeneous Poisson process and the mean arrival rate may change with time. For example, it is much higher during the morning and afternoon rush hours while much lower during night. There is also a moderate peak around noon. In addition, the mean arrival rates for different stations along various bus routes vary with the population density and busyness of the neighborhood of each station. Here we present a brief description of the stochastic process representing the customer arrival process. As stated before we use nonhomogeneous Poisson process with variable intensity (mean). Consider a complete probability space (Ω, Σ, P) and

let $R_0 \equiv [0, \infty)$ and $\mathcal{B}(R_0)$ denote the class of Borel subsets (intervals) of the set R_0. Let $\lambda(t) \geq 0, t \in R_0$, be any locally integrable function, and let $p(\Delta, \lambda(\cdot))$ denote the random number of events occurring during the time interval Δ corresponding to the intensity (mean) function λ. Throughout this paper we use $p(\Delta, \lambda(\cdot))$ to denote any Poisson random measure with the first argument representing any interval (Borel set) and the second one representing mean intensity (arrival rate). The process $\xi(t) \equiv p([0, t), \lambda(\cdot)), t \geq 0$, is called a nonhomogeneous Poisson process if for every positive integer n, and any $\Delta \in \mathcal{B}(R_0)$,

$$P\{p(\Delta, \lambda(\cdot)) = n\} = \exp\left(-\int_\Delta \lambda(t)dt\right) \frac{\left(\int_\Delta \lambda(t)dt\right)^n}{n!}.$$

The dynamics of the overall demand process in the network of the bus routes is given by a system of nonhomogeneous Poisson measures denoted by

$$\left\{p(\Delta, \lambda_{i,j}(\cdot)), j = 1, 2, \cdots, s_i; i = 1, 2, \cdots, n; \Delta \in \mathcal{B}(R_0)\right\}. \tag{6}$$

In order to apply this process to the network, each one of the customer mean arrival rates $\{\lambda_{i,j}(t), t \geq 0\}$ at any station $j \in \{1, 2, \cdots, s_i\}$ along any bus route i will be approximated by an appropriate piecewise constant function. This is quite natural because over a short interval of time, the mean arrival rates $\lambda_{i,j}(t), t \geq 0$, do not change significantly.

Now we are prepared to introduce a measure of customers waiting time. It is clear that it depends on the number of customers arriving at different bus stations, the frequency of city buses running along each line (route), and the importance (or weight) given to each station. Hence, the waiting time at bus station $j \in \{1, 2, \cdots, s_i\}$ along the i-th route during the time period $J_{i,k} \equiv [k\tau_i, (k+1)\tau_i), k \in \mathcal{Z} \equiv \{0, 1, 2, \cdots, N-1\}$ is given by,

$$D_{j,k}(x_i) = \sum_{\xi=0}^{[\frac{\tau_i}{\Delta}]} p(\Delta, \lambda_{i,j}(t_{k\tau_i + \xi\Delta}))\tau_i \tag{7}$$

where Δ is a sufficiently small time segment (say, 5 mins) during which the arrival rate (of customers at any station j in the i-th route) is constant. The mean arrival rates can be obtained from historical (statistical) data of the corresponding station j. $[\frac{\tau_i}{\Delta}]$ is the nearest integer of $\frac{\tau_i}{\Delta}$. So a reasonable measure of waiting time of customers at station j in this route during the whole day is given by the following expression,

$$D_j(x_i) = \sum_{k=0}^{[\frac{t_N}{\tau_i}]-1} D_{j,k}(x_i) \tag{8}$$

So the waiting time of customers for the i-th route during the whole day can be described as,

$$\hat{D}(x_i) = \mathbf{E}\Big\{\sum_{j=1}^{s_i} a_{i,j} D_j(x_i)\Big\} = \mathbf{E}\Big\{\sum_{j=1}^{s_i} \sum_{k=0}^{[\frac{t_N}{\tau_i}]-1} a_{i,j} D_{j,k}(x_i)\Big\}. \tag{9}$$

Therefore, the total (customers) waiting time for all the stations in the network during the entire day is as follows,

$$D(x) = \sum_{i=1}^{n} \hat{D}(x_i) = \mathbf{E}\Big\{\sum_{i=1}^{n}\sum_{j=1}^{s_i}\sum_{k=0}^{[\frac{t_N}{\tau_i}]-1} a_{i,j}D_{j,k}(x_i)\Big\} \qquad (10)$$

where $\mathbf{E}\{\cdot\}$ stands for the expected value of the random variable within the brace, s_i is the total number of stations along the specific i-th route, and $a_{i,j}$ is the importance (or weight) given to the station j along the i-th route which very much depends on the busyness of the station itself. The busier the j-th station is, the larger is $a_{i,j}$. Note that the total waiting time for customers at the station j, given by (8), depends on the Poisson random variable p with the intensity (or mean) function $\lambda_{i,j}$ and this is an integral part of the model. This is what makes the system stochastic. As mentioned earlier, these functions (data) are available from statistical survey of the arrival histories of all the stations under consideration.

3.5. Revenue of the bus company. We assume that the bus ticket price denoted by b is the same in all the city bus routes for all passengers. The number of customers served during the time period $J_{i,k} \equiv [k\tau_i, (k+1)\tau_i)$ at station j in the i-th route is given by,

$$R_{j,k}(x_i) = \sum_{\xi=0}^{[\frac{\tau_i}{\Delta}]} p(\Delta, \lambda_{i,j}(t_{k\tau_i+\xi\Delta})) \qquad (11)$$

Hence, the expected (average) number of customers served at the station j during the whole day is given by the following expression,

$$R_j(x_i) = \mathbf{E}\Big\{\sum_{k=0}^{[\frac{t_N}{\tau_i}]-1} R_{j,k}(x_i)\Big\} \qquad (12)$$

So the average daily revenue of the bus company from this particular route is given by,

$$\hat{R}(x_i) = b\sum_{j=1}^{s_i} R_j(x_i) = b\mathbf{E}\Big\{\sum_{j=1}^{s_i}\sum_{k=0}^{[\frac{t_N}{\tau_i}]-1} R_{j,k}(x_i)\Big\} \qquad (13)$$

Therefore, the expected total daily revenue of the bus company from all the routes it serves is as follows,

$$R(x) = \sum_{i=1}^{n} \hat{R}(x_i) = b\mathbf{E}\Big\{\sum_{i=1}^{n}\sum_{j=1}^{s_i}\sum_{k=0}^{[\frac{t_N}{\tau_i}]-1} R_{j,k}(x_i)\Big\} \qquad (14)$$

4. Formulation of optimization problem.

4.1. Parametric constraints. Define the set U by

$$U \equiv \Big\{x \in R_+^n : x_i \geq 1, \sum_{i=1}^{n} x_i \leq M\Big\}. \qquad (15)$$

Recall that $x = (x_1, x_2, x_3, \cdots, x_n)$ is a n-vector with the i-th entry denoting the number of buses allocated to the i-th route. It is clear that there should be at least one bus allocated to each route ($x_i \geq 1$). The number of buses allocated to any bus route also depends on the number of buses available to the company. Thus we have $\sum_{i=1}^{n} x_i \leq M$. Since the bus company is in charge of serving multiple bus routes (network of city bus routes), it has a great variety of choices to make from the admissible set U. For instance, any choice $x \in U$ can be deployed for the network

of bus routes served by the company. However, the bus company may allocate a selected number of buses $\{x_i^o\}$ to each individual line so that the bus company can maximize its revenue while maintaining the customer satisfaction at an acceptable level. Hence, the objective is to determine the tradeoff between the waiting time, the daily revenue, and the cost of operation by choosing an appropriate number of buses for each individual bus route.

4.2. **Objective functional.** The company's objective is to increase its revenue while reduce customers waiting time, the fuel cost and the operating cost. Using the expressions introduced in section 3, we define the objective (cost) functional $J(x)$ as follows,

$$J(x) = F(x) + P(x) + D(x) - R(x). \tag{16}$$

One may also include the maintenance cost. However, since it is not very significant compared to the daily operating cost, we have excluded it. The objective is to minimize this functional subject to the system dynamics given by (6) and the resource constraint U given by (15).

4.3. **Optimization problem.** The problem is to minimize the cost functional $J(x)$ given by (16) subject to the decision or control constraint set U.

It is believed that the bus company can manage each one of the bus routes in its contract much better (optimally) if the solution of the optimization problem presented here is implemented. The technique presented here can be generally applied to any network of bus routes in any given city. Thus, the solution corresponding to this optimization problem is of great potential to both the city bus company and the people served. This optimization problem can be solved by use of nonlinear integer programming [5]. It is obvious that an optimal policy exists since the decision constraint set U is countable and finite. It is also clear that, for $x^o \in U$ to be optimal it is necessary and sufficient that

$$J(x^o + \Delta x) \geq J(x^o), \; \forall \; \Delta x \text{ such that } (x^o + \Delta x) \in U \text{ and } ||\Delta x|| = 1.$$

5. **Numerical results.** In this section, we conduct a series of numerical simulation experiments for a network consisting of four different bus routes. However the total number of bus routes in the network can be chosen as large as necessary. This will only increase the time of computation. We use the OPTI optimization toolbox [4] based on Matlab to solve the optimization problem stated in the preceding section. The basic idea of the toolbox is the integration of genetic and branch and bound algorithm. Once given an initial guess of the solution, the toolbox will apply the embedded algorithms to search for the optimal solution that maximizes or minimizes a specific objective function. For example, in the numerical experiments we set the initial guess of the number of buses allocated to the four routes as $x = (1, 1, 1, 1)$. As a result, the optimal number of buses allocated to each of the bus routes was found with the corresponding (optimal) cost. In order to further illustrate the feasibility of the proposed stochastic dynamic model and the optimization problem formulated in this paper, we also divided the whole day $[t_0, t_N]$ into three separate time periods and the optimal distribution of city buses was found for each of the time periods.

5.1. **Parameters setup.** As we have stated before, for numerical calculation, the choice of the length of each time interval Δ has to be sufficiently small so that the statistical mean of the Poisson arrival rate does not change. Here we choose a fairly reasonable duration (5 minutes) for each time interval, which results in 288

time intervals in total for the whole day. The parameters chosen for numerical experiments are given in Table 1.

TABLE 1. Parameters for Simulation

Parameter	Value
Length of the i-th route L_i	$L_1 = 24$km, $L_2 = 10$km, $L_3 = 22$km, $L_4 = 15$km
Total number of buses M	10
Number of stations s_i	$s_1 = 16$, $s_2 = 12$, $s_3 = 20$, $s_4 = 14$
Average speed of city buses v_i	$v_1 = 30$, $v_2 = 30$, $v_3 = 35$, $v_4 = 30$
Coefficient of fuel cost $q_{i,k}$	$q_{1,k} = 20$, $q_{2,k} = 20$, $q_{3,k} = 15$, $q_{4,k} = 20$
Weight given to stations $a_{i,j}$	$a_{1,1} = a_{1,2} = a_{1,9} = a_{1,10} = 30$ $a_{1,3} = a_{1,4} = a_{1,11} = a_{1,12} = 60$ $a_{1,5} = a_{1,6} = a_{1,13} = a_{1,14} = 70$ $a_{1,7} = a_{1,8} = a_{1,15} = a_{1,16} = 40$ $a_{2,1} = a_{2,2} = a_{2,3} = 20$ $a_{2,7} = a_{2,8} = a_{2,9} = 50$ $a_{2,4} = a_{2,5} = a_{2,6} = 60$ $a_{2,10} = a_{2,11} = a_{2,12} = 30$ $a_{3,1} = a_{3,2} = a_{3,11} = a_{3,12} = 60$ $a_{3,3} = a_{3,4} = a_{3,13} = a_{3,14} = 80$ $a_{3,5} = a_{3,6} = a_{3,15} = a_{3,16} = 1000$ $a_{3,7} = a_{3,8} = a_{3,17} = a_{3,18} = 70$ $a_{3,9} = a_{3,10} = a_{3,19} = a_{3,20} = 50$ $a_{4,1} = a_{4,2} = a_{4,13} = a_{4,14} = 22$ $a_{4,3} = a_{4,4} = a_{4,11} = a_{4,12} = 52$ $a_{4,5} = a_{4,10} = 65$ $a_{4,6} = a_{4,7} = a_{4,8} = a_{4,9} = 35$
Ticket price b	3
Time interval Δ	5mins

Given the scenario considered here, we assign more weight (importance) to the busier bus stations. This is reflected in the choice of the values of $a_{i,j}$. It is understood that for numerical simulations the relative parameters will have to be set up according to the data and statistics of each specific station.

5.2. **Customer arrival rates.** It is assumed that there are some busier stations. So the mean arrival rates (rate functions) corresponding to these stations are relatively larger than that of other stations. The mean arrival rates together with the corresponding Poisson arrival rates for all the stations in the bus routes of the network are shown in Figure 2, Figure 3, Figure 4 and Figure 5.

Figure 2a and Figure 2c show the customer mean arrival rates for the route 1 over the whole day, for all the stations (station 1 to station 8, and station 9

to station 16, respectively). There is a peak hour period both in the morning and afternoon, as well as a moderate peak around the noon. However, the actual number of customers arriving in any station is random and it follows the nonhomogeneous Poisson process corresponding to the mean intensities discussed above. It is these stochastic processes, as shown in Figure 2b for station 1 to station 8 and Figure 2d for station 9 to station 16, that are used for the numerical experiments. As in Figure 2, Figure 3, Figure 4 and Figure 5 have similar interpretations with regard to route 2, route 3 and route 4, respectively.

(A) Mean arrival rates

(B) Poisson arrival rates

(C) Mean arrival rates

(D) Poisson arrival rates
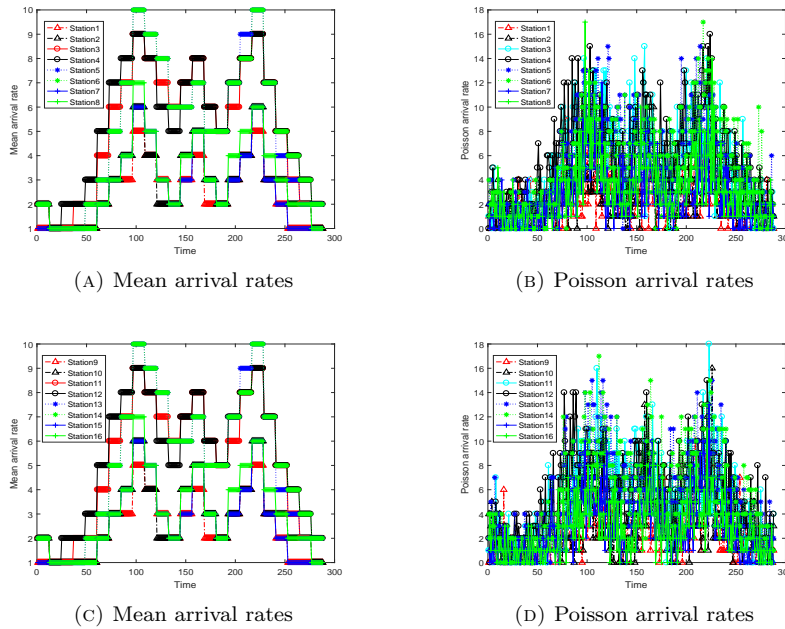
FIGURE 2. Customer arrival rates of station 1 to 16 along route 1

5.3. **Analysis of numerical results.** We have conducted several numerical experiments on the basis of 100 Monte Carlo simulations. The result for each one of the four bus routes over the whole day is shown in Figure 6.

Figure 6 shows the optimal number of buses allocated to each route when optimization process is carried out in terms of each route separately. It can be observed from the graphs that the optimum number of busses to be allocated for each of the four routes are 3, 2, 4 and 2 during the whole day. This indicates that 11 buses in total are required for the network of city bus routes. However, there are only 10 buses available to the company. Therefore, all the city bus routes within the network should be managed optimally at the same time.

For better illustration of the procedure presented in the paper, first suppose there are only route 1 and route 2 in the network. We conducted experiments to determine the optimal number of buses that should be distributed to these two routes (at the same time). The result for this specific scenario is shown in Figure 7.

It can be observed from the graph that the optimum number of buses to be allocated to route 1 and 2 over the whole day are 3 and 2, respectively. This result
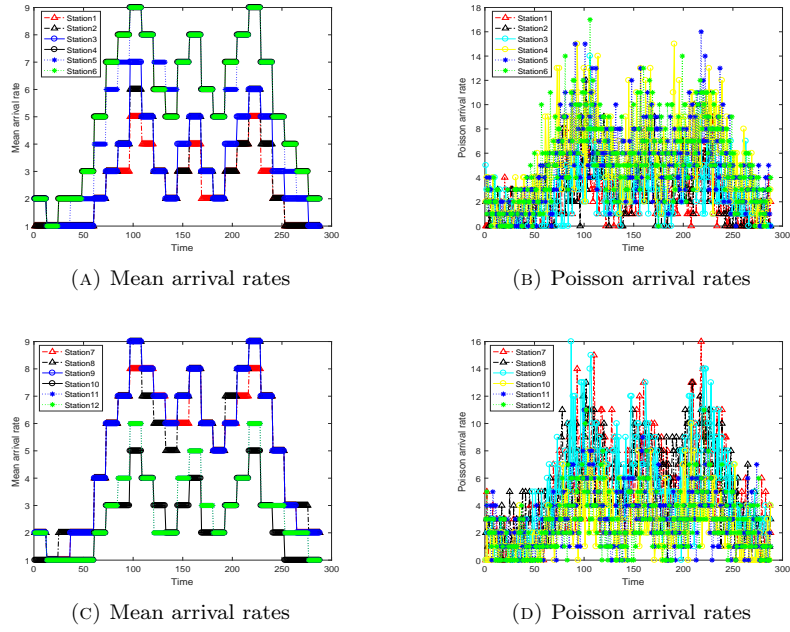
(A) Mean arrival rates

(B) Poisson arrival rates

(C) Mean arrival rates

(D) Poisson arrival rates

FIGURE 3. Customer arrival rates of station 1 to 12 along route 2

(A) Mean arrival rates

(B) Poisson arrival rates

(C) Mean arrival rates

(D) Poisson arrival rates
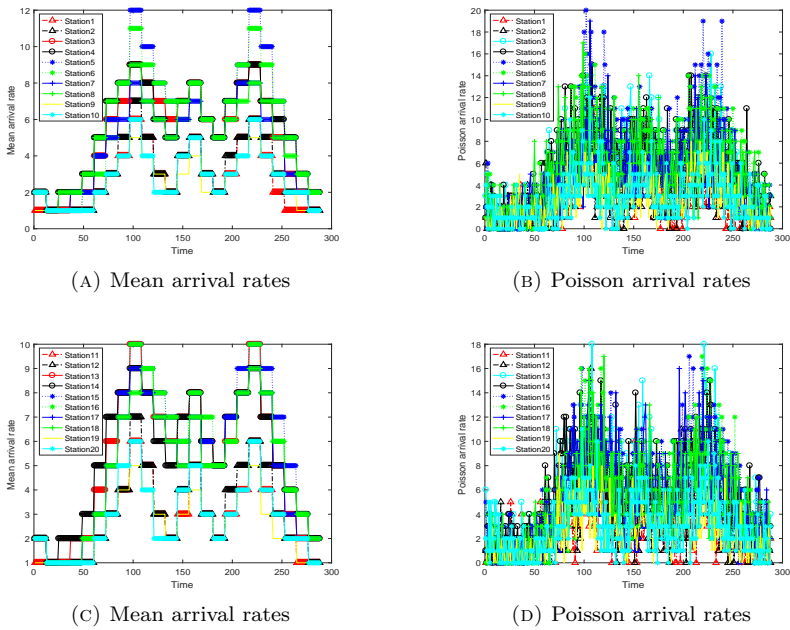
FIGURE 4. Customer arrival rates of station 1 to 20 along route 3

(A) Mean arrival rates

(B) Poisson arrival rates

(C) Mean arrival rates

(D) Poisson arrival rates
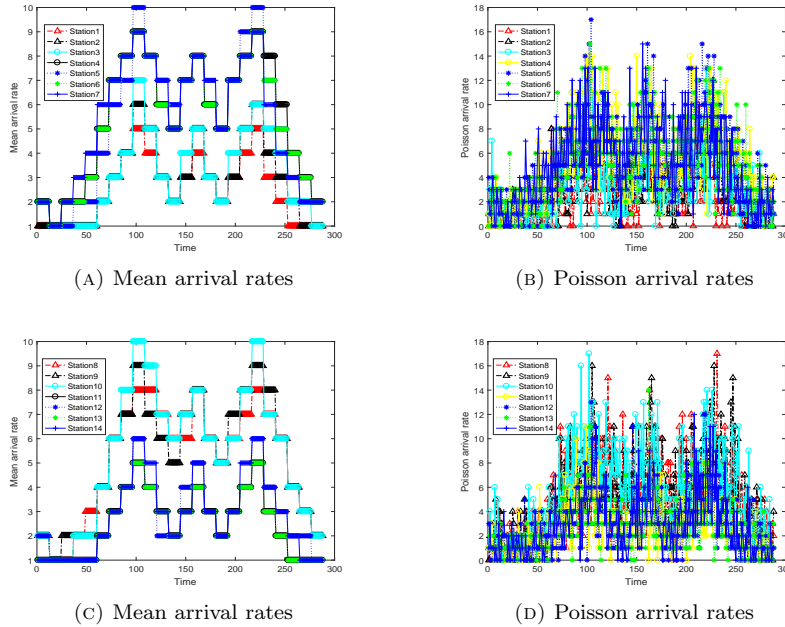
FIGURE 5. Customer arrival rates of station 1 to 14 along route 4

is the same as optimizing these two routes individually since there are only 5 buses required in total (affordable by the company). The optimal cost corresponding to this scenario is 3525963.6333.

In order to determine the optimal number of buses to be allocated to all the four bus routes in the network, we carried out the optimization experiment for the whole day. The result is shown in Table 2. It can be seen that the optimum number of buses distributed to route 1, 2, 3 and 4 are 3, 1, 4 and 2, respectively and the corresponding optimal cost is 7976343.4179.

In order to verify the efficacy of the dynamic model, We also conducted another series of simulation experiments for which the whole day was divided into three parts: 00:00 AM to 6:00 AM, 6:00 AM to 20:00 PM and 20:00 PM to 24:00 PM. This can be easily done by setting $[t_0, t_N]$ as the union of the three disjoint time periods. Note that the choice of the partition of the whole day into three different time periods depends on the city and the bus route itself. Keeping other parameters the same as in the previous simulations, the results for this group of experiments are shown in Table 2.

It can be seen from Table 2 that the optimal choice x^o for these three time periods are [2, 1, 2, 1], [3, 1, 4, 2] and [2, 1, 3, 2], respectively. Based on the objective functional given by the expression (16), the minimum cost corresponding to each time period can be determined as 1317212.4488, 5406920.1899 and 1088617.3315, respectively. Adding these we obtain the total cost for the whole day as 7812749.9702, which is less than the minimum cost 7976343.4179 obtained by optimizing directly for the whole day as seen in the previous scenario. It is clear that we have different minimum costs for different scenarios. This implies that the (minimum) cost can be reduced further by dividing the entire day into several intervals based on the

(A) Route 1

(B) Route 2

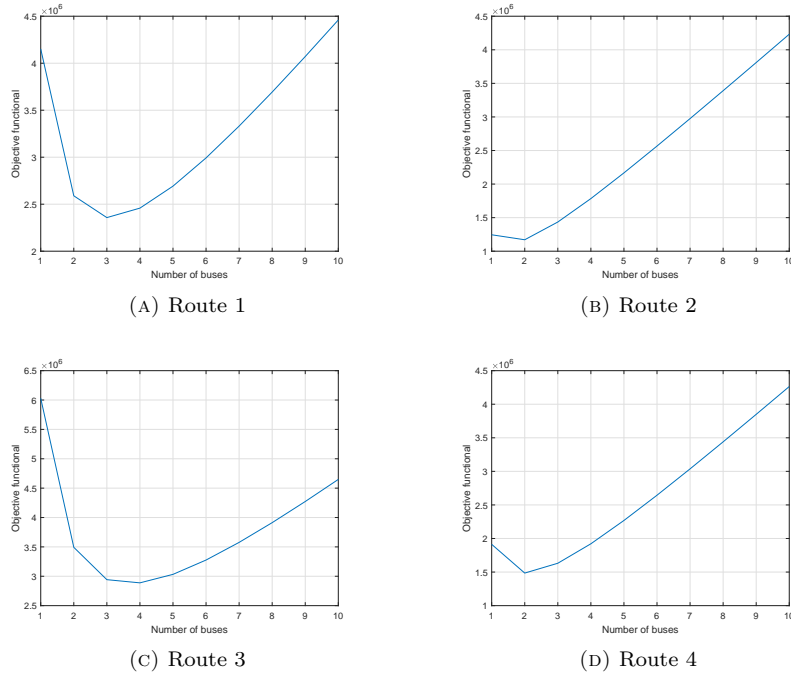(C) Route 3

(D) Route 4

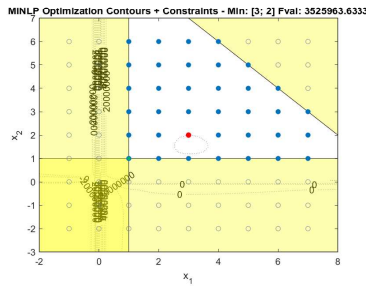FIGURE 6. Simulation result for each separate route over the whole day.

FIGURE 7. Result for route 1 and 2 over the whole day.

TABLE 2. Simulation Result for the Network of City Bus Routes

Time	Optimal control x^o	Optimal cost
Whole day	[3, 1, 4, 2]	7976343.4179
00:00 AM to 6:00 AM	[2, 1, 2, 1]	1317212.4488
6:00 AM to 20:00 PM	[3, 1, 4, 2]	5406920.1899
20:00 PM to 24:00 PM	[2, 1, 3, 2]	1088617.3315

periods of customer demand depending on the long-term statistics. It is clear that
this procedure may require frequent rescheduling.

6. **Conclusion.** A stochastic dynamic model for the network of city bus routes subject to resource constraints has been proposed along with a method for determining the optimal strategy designed to maximize the company's revenue, and minimize the fuel cost and the operating cost as well as customers waiting time. The proposed system was simulated with customer arrival rates expressed by nonhomogeneous Poisson processes. The optimization results giving the optimal number of buses required to serve the network were found. Two sets of results are given, one involving optimization over the whole day and another involving optimization over selected periods of the day. This dynamic model is sufficiently general and can be easily employed for different cities. However, in order to apply this technique to any given city, it is necessary to collect the statistics (data) of the customer mean arrival rates for each station in the network. It is expected that the proposed scheme will improve the overall quality of service provided by the bus company and it will also increase its revenue.

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